THE DEPTH OF A HYPERSUBSTITUTION

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Abstract. For given depths of the terms $s, t_1, \cdots, t_n$ a formula will be proved to calculate the depth of the composed term $s(t_1, \cdots, t_n)$ and if σ is a hypersubstitution and t is a term we derive a formula for the depth of $\hat{\sigma}[t]$.

1. Introduction

At first we remember of the following definition of terms. Let $X = \{x_1, \cdots, x_n, \cdots\}$ be any countably infinite (standard) alphabet of variables and let $X_n = \{x_1, \cdots, x_n\}$ be an n-element alphabet. Let $(f_i)_{i \in I}$ be an indexed set which is disjoint from X. Each f_i is called an n_i-ary operation symbol where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The function τ or the sequence of values of τ, written as $(n_i)_{i \in I}$, is called a type. An n-ary term of type τ is defined inductively as follows:

(i) The variables $x_1, \cdots, x_n$ are n-ary terms.
(ii) If $t_1, \cdots, t_m$ are n-ary terms and if f_i is an n_i-ary operation symbol then $f_i(t_1, \cdots, t_{n_i})$ is an n-ary term.
(iii) Let $W_\tau(X_n)$ be the smallest set which contains $x_1, \cdots, x_n$ and is closed under finite application of (ii). Every $t \in W_\tau(X_n)$ is called an n-ary term of type τ.

We remark that by this definition every n-ary term is also $(n+1)$-ary. The set $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type τ.

Usually one has a third set $\overline{A}$, called set of constants, with $\overline{A} \cap X = \emptyset$ and $\{f_i | i \in I\} \cap (\overline{A} \cup X) = \emptyset$. Then polynomials of type τ over $\overline{A}$ are defined in a similar way adding a condition which says that constants are polynomials. Constants can also be defined by nullary operation symbols assuming that the indexed set $(f_i)_{i \in I}$ of operation symbols includes also nullary operation symbols ($n_i = 0$). We mention also that in the case of finite sets of operation symbols instead of polynomials one speaks of trees. Trees can be regarded as connected graphs without cycles. Trees have many applications in Computer Science, in Linguistic and in other fields. Ordered binary decision diagrams (OBDD's) are trees in the language of Boolean algebras. Trees can be used to visualize the structure of computer programmes. For all these applications it is important to measure the complexity of a tree or of a term. The concept of the depth of a term (or of the height of a tree) is a well-known

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complexity measure. The method of algebraic induction which is very often used, is based on the depth of terms or polynomials.

Definition 1.1. *Let $t \in W_\tau(X)$ be a term.*

(i) *If $t = x \in X$ then $Depth(t) := 0$.*
(ii) *If $t = f_i(t_1, \cdots, t_{n_i})$ then*
$Depth(t) := max\{Depth(t_1), \cdots, Depth(t_{n_i})\} + 1$.

The depth of a polynomial is defined in a similar way where $Depth(\overline{a}) := 0$ if $\overline{a}$ is a constant from $\overline{A}$.

In the case of a tree usually one speaks of the height of the tree.

Our goal is to describe the behavior of the depth under some mappings defined on sets of terms. We select two mappings which play an important role in Universal Algebra but also in Computer Science.

The first mapping is called composition of terms and is defined in the following inductive way:
Let $s \in W_\tau(X_n)$ and let $t_1, \cdots, t_n \in W_\tau(X_m)$. Then we define

$$S_m^n : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$$

by the following steps:

(i) If $s = x_i, 1 \leq i \leq n$ then $S_m^n(s, t_1, \cdots, t_n) := t_i$.
(ii) If $s = f(s_1, \cdots, s_r)$ and $s_1, \cdots, s_r \in W_\tau(X_n)$ then
$S_m^n(s, t_1, \cdots, t_n) := f(S_m^n(s_1, t_1, \cdots, t_n), \cdots, S_m^n(s_r, t_1, \cdots, t_n))$
$= f(s_1(t_1, \cdots, t_n), \cdots, s_r(t_1, \cdots, t_n))$.

We remark that the heterogeneous (multibased) algebra

$$((W_\tau(X_n))_{n \in \mathbb{N}^+}, (S_m^n)_{m,n \in \mathbb{N}^+}, (x_i)_{i \leq n \in \mathbb{N}^+})$$

with $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ with variables as nullary operations is called full term clone of type τ. (There are only technical reasons not to consider nullary terms, but one can define a similar structure for polynomials as well.) For proofs by induction it is very advantageous to use the operations S_m^n to describe the superposition of terms.

The second kind of mappings is called hypersubstitution of type τ, for short, hypersubstitution. Hypersubstitutions play an important role in the theory of hyperidentities and solid varieties which is a very fast-developing modern algebraic theory with applications in Theoretical Computer Science (hyper-tree automata) and in Logic (fragment of second order logic).

Hypersubstitutions σ of type τ are defined by $\sigma(f_i) = t$ where t is an n_i-ary term for all n_i-ary operation symbols f_i. Those mappings can be extended to mappings $\hat{\sigma}$ defined on sets of terms by the following steps:

(i) $\hat{\sigma}[x] := x$ if $x \in X$ is a variable.
(ii) $\hat{\sigma}[f_i(t_1, \cdots, t_{n_i})] := S_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_{n_i}])$ for composed terms $f_i(t_1, \cdots, t_{n_i})$.

This definition shows that the mappings $\hat{\sigma}$ are endomorphisms of the full term clone (as heterogeneous algebra). In the case of trees the mappings $\hat{\sigma}$ are special types of so-called alphabetic tree homomorphisms ([4, 1]) and preserve the recognizability

of a forest (set of trees) by a tree automaton.
Having a look on trees the formula which we will derive for the depth of composed trees becomes quite clear and simple. Nevertheless we will give full proofs, mostly by induction since terms can be countably infinite. For the depth of an arbitrary hypersubstitution we obtained a more complicated formula and its proof is far from beeing trivial.

2. Full Terms

Our first aim is to calculate $Depth(S^n_m(s,t_1,\cdots,t_n))$ if $Depth(s), Depth(t_1),\cdots,Depth(t_n)$ are known. The following example shows that $Depth(S^n_m(s,t_1,\cdots,t_n))$ depends not only on the depths of the inputs but also on the special structure of the term $S^n_m(s,t_1,\cdots,t_n)$.

Consider the type $\tau=(2)$ with f as binary operation symbol and the terms

$$t_1=f(x_1,f(x_1,x_2)), t_2=f(x_2,x_1) \quad \text{and} \quad s_1=f(f(x_2,x_2),x_1).$$

Then we have

$$Depth(t_1)=2, Depth(t_2)=1, Depth(s_1)=2 \quad \text{and}$$

$$S^2_2(s_1,t_1,t_2)=s_1(t_1,t_2)=f(f(f(x_2,x_1),f(x_2,x_1)),f(x_1,f(x_1,x_2))), \quad \text{and}$$

$$Depth(S^2_2(s_1,t_1,t_2))=3.$$

Now instead of s_1 we take the term $s_2=f(f(x_1,x_1),x_2)$ with $Depth(s_2)=2$ and obtain

$$S^2_2(s_2,t_1,t_2)=f(f(f(x_1,f(x_1,x_2)),f(x_1,f(x_1,x_2))),f(x_2,x_1))$$

with $Depth(S^2_2(s_2,t_1,t_2))=4$. That means, $Depth(S^n_m(s,t_1,\cdots,t_n))$ depends on the particular structure of the terms. But this is not always the case. If, for instance, $s=x_i, 1\le i\le n$ is a variable then $Depth(s)=0$ and $Depth(S^n_m(s,t_1,\cdots,t_n))=Depth(t_i)$.

Now we consider the following kind of terms, called full terms:

Definition 2.1. (i) *If f_i is an n_i-ary operation symbol and if $s:\{1,\cdots,n_i\}\to\{1,\cdots,n_i\}$ is a permutation then $f_i(x_{s(1)},\cdots,x_{s(n_i)})$ is a full term.*
(ii) *If f_j is an n_j-ary operation symbol and if $t_1,\cdots,t_{n_j}$ are full terms then $f_j(t_1,\cdots,t_{n_j})$ is a full term.*

By $W^f_\tau(X)$ we denote the set of all full terms of type τ. It is easy to see that the set $W^f_\tau(X)$ of all full terms is closed under composition.

Lemma 2.2. *Let $s\in W^f_\tau(X_n)$ and let $t_1,\cdots,t_n\in W^f_\tau(X_m), 1\le n,m\in I\!N$ be full terms. Then $S^n_m(s,t_1,\cdots,t_n)$ is also a full term.*

Proof. We give a proof by induction on the complexity ($Depth$) of a term s. If $s=f(x_{s(1)},\cdots,x_{s(n)})$ where s is a permutation on the set $\{1,\cdots,n\}$ then $S^n_m(s,t_1,\cdots,t_n)=f(t_{s(1)},\cdots,t_{s(n)})$ is a full term by Definition 2.1(ii).

Now, let $s=f(s_1,\cdots,s_r), s_1,\cdots,s_r\in W_\tau(X_n)$ and let $s\in W^f_\tau(X_n)$, $t_1,\cdots,t_n\in W^f_\tau(X_m)$. We assume that $S^n_m(s_j,t_1,\cdots,t_n)$ are full terms for all $1\le j\le r$. Then

$$S^n_m(s,t_1,\cdots,t_n)=f(S^n_m(s_1,t_1,\cdots,t_n),\cdots,S^n_m(s_r,t_1,\cdots,t_n))$$

and by Definition 2.1 (ii) $S^n_m(s,t_1,\cdots,t_n)$ is a full term. ∎

For full terms we have:

Theorem 2.3. *Let $s \in W_\tau^f(X_n)$ and assume that $t_1, \cdots, t_n \in W_\tau(X_m), 1 \leq m, n \in I\!N$ and that $\tau = (n, \cdots, n)$, i.e., all operation symbols have the same arity. Then*

$$Depth(S_m^n(s, t_1, \cdots, t_n)) = max\{Depth(t_1), \cdots, Depth(t_n)\} + Depth(s).$$

Proof. We give a proof by induction on $Depth(s)$. If $Depth(s) = 1$ then $s = f(x_{s(1)}, \cdots, x_{s(n)})$ for an n-ary operation symbol f and a permutation $s : \{1, \cdots, n\} \to \{1, \cdots, n\}$. There follows that

$$Depth(S_m^n(s, t_1, \cdots, t_n)) = Depth(f(t_{s(1)}, \cdots, t_{s(n)}))$$

$$= max\{Depth(t_{s(1)}), \cdots, Depth(t_{s(n)})\} + 1 = max\{Depth(t_1), \cdots, Depth(t_n)\} + 1.$$

Assume that the formula is satisfied for $s_1, \cdots, s_r$ and assume that $s = f(s_1, \cdots, s_r)$. Note that if $s \in W_\tau^f(X_n)$ then also $s_1, \cdots, s_r \in W_\tau^f(X_n)$, i.e. one can assume that

$$Depth(S_m^n(s_j, t_1, \cdots, t_n)) = max\{Depth(t_1), \cdots, Depth(t_n)\} + Depth(s_j)$$

for $1 \leq j \leq r$. Then we have

$$S_m^n(s, t_1, \cdots, t_n) =$$

$$S_m^n(f(s_1, \cdots, s_r), t_1, \cdots, t_n) = f(S_m^n(s_1, t_1, \cdots, t_n), \cdots, S_m^n(s_r, t_1, \cdots, t_n))$$

and

$$Depth(S_m^n(s, t_1, \cdots, t_n))$$

$$= max\{Depth(S_m^n(s_1, t_1, \cdots, t_n)), \cdots, Depth(S_m^n(s_r, t_1, \cdots, t_n))\} + 1$$

$$= max\{max\{Depth(t_1), \cdots, Depth(t_n)\} + Depth(s_1),$$

$$\cdots, max\{Depth(t_1), \cdots, Depth(t_n)\} + Depth(s_r)\} + 1$$

$$= max\{Depth(t_1), \cdots, Depth(t_n)\} + max\{Depth(s_1), \cdots, Depth(s_r)\} + 1$$

$$= max\{Depth(t_1), \cdots, Depth(t_n)\} + Depth(s)$$

since $s = f(s_1, \cdots, s_r)$. ■

3. The Depth of a Term with Respect to a Variable

To derive a formula for the depth of the superposition of arbitrary terms we define at first the depth of a term with respect to a variable. Let $t \in W_\tau(X_n)$ be an n-ary term and let $var(t)$ be the set of all variables occurring in the term t.

Definition 3.1.

(i) *If $t = x_k, 1 \leq k \leq n$, then $Depth_l(t) := 0$ for all $1 \leq l \leq n$.*

(ii) *If $t = f_i(t_1, \cdots, t_{n_i})$ where f_i is n_i-ary and if we assume that $Depth_l(t_j), 1 \leq j \leq n_i, 1 \leq l \leq n$ are already defined then for all $l, 1 \leq l \leq n$ we define*

$$Depth_l(t) := \begin{cases} 0, & \text{if } x_l \notin var(t) \\ max\{Depth_l(t_j) | 0 \leq j \leq n_i, x_l \in var(t_j)\} + 1, & \text{otherwise.} \end{cases}$$

Example 3.2. *Consider $I = \{1,2\}$ and the type $\tau = (2,3)$.*

For the term $t_1 = f_2(f_1(x_1,x_1), f_1(x_1,x_2), x_3)$ we have $Depth_1(t_1) = 2, Depth_2(t_1) = 2, Depth_3(t_1) = 1$ and $Depth(t_1) = 2$.

For the term $t_2 = f_1(f_2(x_1,x_1,x_2),x_1)$ we have $Depth_1(t_2) = 2, Depth_2(t_2) = 2, Depth_3(t_2) = 0$ and $Depth(t_2) = 2$.

For $t_3 = f_1(f_2(x_1,x_3,x_3),x_1)$ one has $Depth_1(t_3) = 2, Depth_2(t_3) = 0, Depth_3(t_3) = 2$.

For $s = f_2(f_1(x_1,x_2),x_2,x_3)$ one obtains $Depth_1(s) = 2, Depth_2(s) = 2$ and $Depth_3(s) = 1$.

Consider $S_3^3(s,t_1,t_2,t_3)$. Then it is easy to calculate that $Depth(S_3^3(s,t_1,t_2,t_3)) = 4$. This is equal to

$$max\{Depth_1(s) + Depth(t_1), Depth_2(s) + Depth(t_2), Depth_3(s) + Depth(t_3)\}.$$

More generally, we prove

Theorem 3.3. *Let $s \in W_\tau(X_n), t_1, \cdots, t_n \in W_\tau(X_m)$. Then*

$$Depth(S_m^n(s,t_1,\cdots,t_n)) = max\{Depth_j(s) + Depth(t_j) | 1 \le j \le n, x_j \in var(s)\}.$$

Proof. We prove the formula by induction on $Depth(s)$. If $Depth(s) = 0$ then there exists a natural number $k \in \{1,\cdots,n\}$ such that $s = x_k$ and then $S_m^n(s,t_1,\cdots,t_n) = t_k$ and thus $Depth(S_m^n(s,t_1,\cdots,t_n)) = Depth(t_k)$ and $max\{Depth_j(s) + Depth(t_j) | 1 \le j \le n, x_j \in var(s)\} = Depth(x_k) + Depth(t_k) = 0 + Depth(t_k) = Depth(t_k)$.

Assume now that the formula is satisfied for $s_1,\cdots,s_r$ and assume that $s = f(s_1,\cdots,s_r)$.
Then

$$S_m^n(s,t_1,\cdots,t_n) = f(S_m^n(s_1,t_1,\cdots,t_n),\cdots,S_m^n(s_r,t_1,\cdots,t_n))$$

and

$$\begin{aligned} &Depth(S_m^n(s,t_1,\cdots,t_n)) \\ &= max\{Depth(S_m^n(s_1,t_1,\cdots,t_n)),\cdots,Depth(S_m^n(s_r,t_1,\cdots,t_n))\} + 1 \\ &= max\{max\{Depth_j(s_1) + Depth(t_j) | 1 \le j \le n, x_j \in var(s_1)\},\cdots, \\ &\quad max\{Depth_j(s_r) + Depth(t_j) | 1 \le j \le n, x_j \in var(s_r)\}\} + 1 \\ &= max\{max\{Depth_j(s_k) | 1 \le k \le r, x_j \in var(s_k)\} + 1 + Depth(t_j) | 1 \le j \le n, \\ &\quad x_j \in \bigcup\{var(s_k) | 1 \le k \le r\}\} \\ &= max\{max\{Depth_j(s_k) | 1 \le k \le r, x_j \in var(s_k)\} + 1 + Depth(t_j) | 1 \le j \le n, \\ &\quad x_j \in var(s)\} = max\{Depth_j(s) + Depth(t_j) | 1 \le j \le n, x_j \in var(s)\}. \end{aligned}$$

■

It is clear that the depth of a term t is the maximum of all $Depth_j(t)$ for $x_j \in var(t)$, i.e.,

Lemma 3.4. *If $t \in W_\tau(X_n), 1 \le n \in \mathbb{N}$ then*

$$Depth(t) = max\{Depth_j(t) | 1 \le j \le n, x_j \in var(t)\}.$$

Proof. We will give a proof by induction on $Depth(t)$. If $Depth(t) = 0$ then $t = x \in X$ is a variable and $Depth_j(t) = 0$ for $1 \leq j \leq n$. Hence $max\{Depth_j(t)|1 \leq j \leq n, x_j \in var(t)\} = 0 = Depth(t)$.

Assume the lemma is satisfied for $s_1, \cdots, s_r$ and that $t = f(s_1, \cdots, s_r)$. Then
$Depth(t)$
$= max\{Depth(s_k)|k \in \{1, \cdots, r\}\} + 1$
$= max\{max\{Depth_j(s_k)|1 \leq j \leq n, x_j \in var(s_k)\}|1 \leq k \leq r\} + 1$
$= max\{max\{Depth_j(s_k)|1 \leq k \leq r, x_j \in var(s_k)\} + 1|1 \leq j \leq n, x_j \in var(t)\}$
$= max\{Depth_j(t)|1 \leq j \leq n, x_j \in var(t)\}$. ■

4. Full Hypersubstitutions

In section 1 we have already introduced the concept of a hypersubstitution. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. On $Hyp(\tau)$ by $(\sigma_1 \circ_h \sigma_2)(f_i) := \hat{\sigma}_1[\sigma_2(f_i)]$ for all operation symbols f_i a binary operation can be defined. Then $Hyp(\tau)$ together with the identity hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) := f_i(x_1, \cdots, x_{n_i})$ forms a monoid. If V is a variety of algebras of type τ then V is called solid if for every identity $s \approx t$ in V and every $\sigma \in Hyp(\tau)$ the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are satisfied as identities in V. All solid varieties of type τ form a complete sublattice of the lattice of all varieties of type τ. If M is a submonoid of $Hyp(\tau)$ one can define M-solid varieties of type τ. The class of all M-solid varieties of type τ forms also a complete lattice and the collection of all solid varieties is a complete sublattice of the lattice of all M-solid varieties of type τ. More generally, if $M_1 \subseteq M_2$ for two submonoids M_1, M_2 of $Hyp(\tau)$ then the collection of all M_2-solid varieties of type τ forms a complete sublattice of the lattice of all M_1-solid varieties of type τ.

For more background on hyperidentities, hypersubstitutions and solid varieties see [2, 3].

Now we consider a special class of hypersubstitutions of type τ.

Definition 4.1. *A hypersubstitution is called full if $\sigma(f_i) \in W_\tau^f(X_{n_i})$ for all $i \in I$. By $Hyp^f(\tau)$ we denote the set of all full hypersubstitutions of type τ.*

Lemma 4.2. *The set $Hyp^f(\tau)$ forms a submonoid of the monoid $Hyp(\tau)$ of all hypersubstitutions of type τ.*

Proof. Since the terms $\sigma_{id}(f_i) = f_i(x_1, \cdots, x_{n_i})$ are full terms for every $i \in I$ the identity hypersubstitution is full. Assume that $\sigma_1, \sigma_2 \in Hyp^f(\tau)$. We want to prove that $(\sigma_1 \circ_h \sigma_2)(f_i)$ are full terms for every $i \in I$. Since $\sigma_2(f_i)$ is a full term, by definition of full terms there exists an operation symbol f_j such that $\sigma_2(f_i) = f_j(x_{s(1)}, \cdots, x_{s(x_{n_j})})$ for a permutation $s : \{x_1, \cdots, x_{n_j}\} \to \{x_1, \cdots, x_{n_j}\}$ or full terms $t_1, \cdots t_{n_j}$ with $\sigma_2(f_i) = f_j(t_1, \cdots, t_{n_j})$. In the first case we have

$$(\sigma_1 \circ_h \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)] = \sigma_1(f_j)(x_{s(1)}, \cdots, x_{s(n_j)}).$$

Since σ_1 is a full hypersubstitution, the term $\sigma_1(f_j)$ is a full term and then for every permutation $s : \{1, \cdots, n_j\} \to \{1, \cdots, n_j\}$ the term $\sigma_1(f_j)(x_{s(1)}, \cdots, x_{s(n_j)})$ is also full. In the second case one obtains

$$(\sigma_1 \circ_h \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)] = \hat{\sigma}_1[f_j(t_1, \cdots, t_{n_j})] = \sigma_1(f_j)(\hat{\sigma}_1[t_1], \cdots, \hat{\sigma}_1[t_{n_j}]).$$

Here $\sigma_1(f_j)$ is a full term. We show that $\hat{\sigma}[t_k]$ are also full terms for all $k \in \{1, \cdots, n_j\}$. In fact, if $t_k = f_\mu(x_{s(1)}, \cdots, x_{s(n_\mu)})$ then $\hat{\sigma}_1[t_k] = \sigma_1(f_\mu)(x_{s(1)}, \cdots, x_{s(n_\mu)})$ where $s : \{1, \cdots, n_\mu\} \to \{1, \cdots, n_\mu\}$ is a permutation. Since σ_1 is a full hypersubstitution, the term $\sigma_1(f_\mu)$ is a full term and then for every permutation $s : \{1, \cdots, n_\mu\} \to \{1, \cdots, n_\mu\}$ the term $\sigma_1(f_\mu)(x_{s(1)}, \cdots, x_{s(n_\mu)})$ is also full. If $t_k = f_\mu(t_{11}, \cdots, t_{1n_\mu})$ and assume that $\hat{\sigma}_1[t_{1j}], 1 \leq j \leq n_\mu$ are full then $\hat{\sigma}_1[t_k] = \sigma_1(f_\mu)(\hat{\sigma}[t_{11}], \cdots, \hat{\sigma}[t_{1n_\mu}])$ is also full by Definition 2.1 (ii). But then by Lemma 2.2 $\sigma_1(f_j)(\hat{\sigma}[t_1], \cdots \hat{\sigma}[t_{n_j}])$ is also a full term and $\sigma_1 \circ_h \sigma_2 \in Hyp^f(\tau)$. ■

For a given variety V of type τ one can determine all subvarieties of V which are $Hyp^f(V)$-solid. Consider as an example the variety of all bands (idempotent semigroups). A hypersubstitution σ of type τ is called a regular hypersubstitution of type τ if $var(\sigma(f_i)) = \{x_1, \cdots, x_{n_i}\}$ for all $i \in I$. The collection $Reg(\tau)$ of all regular hypersubstitutions of type τ forms also a submonoid $Reg(\tau)$ of $Hyp(\tau)$ and a variety V of type τ is called regular-solid if it is $Reg(\tau)$-solid ([3]).

Then we get

Proposition 4.3. *A variety V of bands is $Hyp^f(2)$-solid iff V is regular-solid.*

Proof. By definition every full hypersubstitution is regular, i.e., $Hyp^f(2) \subseteq Reg(2)$. Therefore, if V is regular-solid then it is also $Hyp^f(2)$-solid. It remains to show that a $Hyp^f(2)$-solid variety of bands is regular-solid. Let $\sigma_t \in Reg(2)$ where $t \in W_2(X_2)$. (Here σ_t is the hypersubstitution which maps the binary operation symbol f to the binary term t.) In t both variables x_1 and x_2 occur and there are exactly the following cases:
a) t starts with x_1 and ends with x_2,
b) t starts with x_2 and ends with x_1,
c) t starts and ends with x_1,
d) t starts and ends with x_2.

In the first case we have $t \approx x_1x_2$, in the second case there holds $t \approx x_2x_1$, in the third case one obtains $t \approx x_1x_2x_2x_1$ and in the fourth case $t \approx x_2x_1x_1x_2$. All four hypersubstitutions are full and since V is $Hyp^f(2)$-solid it is also regular-solid. ■

If the depth of $\sigma(f_i)$ for a hypersubstitution σ for all $i \in I$ is known and if $Depth(t)$ for $t \in W_\tau(X)$ is known we want to know what $Depth(\hat{\sigma}[t])$ is. It is quite natural to define the depth of a hypersubstitution σ in the following way:

Definition 4.4. *Let σ be a hypersubstitution of type τ then*

$$Depth(\sigma) := max\{Depth(\sigma(f_i))|i \in I\}.$$

Clearly, for the type $\tau = (n)$ we have $Depth(\sigma) = Depth(\sigma(f))$. Then we obtain the following result:

Corollary 4.5. *Let $t \in W_\tau^f(X_n), \sigma \in Hyp^f(\tau)$ and $\tau = (n)$ for a natural number $n \geq 1$. Then*

$$Depth(\hat{\sigma}[t]) = Depth(\sigma(f))Depth(t).$$

Proof. We give a proof by induction on $Depth(t)$. If $Depth(t) = 1$ then $t = f(x_{s(1)}, \cdots, x_{s(n)})$ for a permutation $s : \{1, \cdots, n\} \to \{1, \cdots, n\}$ and

$\hat{\sigma}[t] = \sigma(f)(x_{s(1)}, \cdots, x_{s(n)}) = S^n_n(\sigma(f), x_{s(1)}, \cdots, x_{s(n)})$. Therefore

$$\begin{aligned}
Depth(\hat{\sigma}[t]) &= Depth(S^n_n(\sigma(f), x_{s(1)}, \cdots, x_{s(n)})) \\
&= max\{Depth(x_{s(1)}), \cdots, Depth(x_{s(n)})\} + Depth(\sigma(f)) \\
&= 0 + Depth(\sigma(f)) \\
&= Depth(\sigma(f)) = Depth(\sigma(f))Depth(t).
\end{aligned}$$

Assume that $Depth(\hat{\sigma}[s_k])) = Depth(\sigma(f))Depth(s_k)$ for $1 \leq k \leq n$ and $t = f(s_1, \cdots, s_n)$. Then

$$\begin{aligned}
Depth(\hat{\sigma}[t]) &= Depth(S^n_n(\sigma(f), \hat{\sigma}[s_1], \cdots, \hat{\sigma}[s_n])) \\
&= Depth((\sigma(f)) + max\{Depth(\hat{\sigma}[s_k])|1 \leq k \leq n\} \\
&= Depth((\sigma(f)) + max\{Depth(\sigma(f))Depth(s_k)|1 \leq k \leq n\} \\
&= Depth((\sigma(f)) + Depth(\sigma(f))max\{Depth(s_k)|1 \leq k \leq n\} \\
&= Depth((\sigma(f))(1 + max\{Depth(s_k)|1 \leq k \leq n\}) \\
&= Depth(\sigma(f))Depth(t).
\end{aligned}$$

■

As a consequence of Corollary 4.5 we have:

Corollary 4.6. *The function $Depth^f : Hyp^f(\tau) \to I\!N^+$ with $\sigma \mapsto Depth(\sigma)$ defines a homomorphism from the monoid $(Hyp^f(\tau); \circ_h, \sigma_{id})$ onto the monoid $(I\!N^+; \cdot, 1)$.*

Proof. The mapping $Depth^f$ is well-defined and surjective since to every natural number $n \geq 1$ there is a full term t with $Depth(t) = n$. Further we have

$$\begin{aligned}
Depth^f(\sigma_{id}) &= Depth(\sigma_{id}(f)) \\
&= Depth(f(x_1, \cdots, x_n)) = 1
\end{aligned}$$

and

$$\begin{aligned}
Depth^f(\sigma_1 \circ_h \sigma_2) &= Depth((\sigma_1 \circ_h \sigma_2)(f)) \\
&= Depth(\hat{\sigma}_1[\sigma_2(f)]) \\
&= Depth(\sigma_1(f))Depth(\sigma_2(f)) \\
&= Depth^f(\sigma_1)Depth^f(\sigma_2).
\end{aligned}$$

■

5. The Depth of an Arbitrary Hypersubstitution

To derive a formula for $Depth(\hat{\sigma}[t])$ we introduce the following notation:
The yield $yd(t)$ of the term t is defined inductively as follows:

(i) $yd(x) = x$ for all $x \in X$.
(ii) If $t = f_i(t_1, \cdots, t_{n_i})$ then $yd(t) = yd(t_1) \cdots yd(t_{n_i})$.

That means, the yield of a term t is the semigroup word obtained from t by cancellation of all operation symbols, brackets, and commas.

The length $\ell(t)$ of the term t is the number of variables occurring in t.

Assume that $t \in W_\tau(X)$ has the length $\ell(t) = n$ and that $i \in \{1, \cdots, n\}$. Let $yd(t) = u_1 \cdots u_{\ell(t)}$, where $u_1 \cdots u_{\ell(t)}$ denote certain variables. Then we define terms $A_k(i,t)$ for $0 \leq k \in I\!N$ and for $1 \leq i \leq \ell(t)$ in the following inductive way:

(i) $A_0(i,t) := u_i$,
(ii) $A'_{\ell(t)} := t$,
$A'_k := t_r$, if $A'_{k+1} := f_j(t_1, \ldots t_{n_j})$ and u_i is contained as a subterm in t_r $(1 \leq r \leq n_j)$ (where r is uniquely determined),
$A'_k := u_i$, if $A'_{k+1} := u_i$ for $0 \leq k < \ell(t)$,
(iii) Let r be the greatest integer with $A'_r = u_i$. Then we define
$A_i := A'_{i+r}$ for $0 \leq i < \ell(t) - r$ and

$A_i := t$ for $\ell(t) - r < k \in I\!N$.

By $\beta(i,t)$ we denote the least natural number k with $A_k(i,t) = t$. Let $\sigma \in Hyp(\tau)$. Then we define:

(i) $B_0(\sigma, i, t) := 0$,

(ii) $B_k(\sigma, i, t) := Depth_a(\sigma(f_j))$ if $1 \leq k \leq \beta(i,t)$ and $a \in \{1, 2, \cdots, n_j\}$ is determined by the property that $A_{k-1}(i,t)$ is at the a-th place in the term $A_k(i,t) = f_j(t_1, \cdots, t_{n_j})$.

Finally we define $B(\sigma, i, t) := \sum\limits_{k=0}^{\beta(i,t)} B_k(\sigma, i, t)$ and

$B(\sigma, t) := max(\{B(\sigma, i, t) | 1 \leq i \leq \ell(t), B_{\beta(\sigma,i,t)}(i,t) \neq 0\} \cup \{0\})$

Then we have

Theorem 5.1. *Let $t \in W_\tau(X)$ and $\sigma \in Hyp(\tau)$. Then $Depth(\hat{\sigma}[t]) = B(\sigma, t)$.*

Proof. We will give a proof by induction on $Depth(t)$. If $Depth(t) = 0$ then $t \in X, \ell(t) = 1, A_0(1,t) = t$, and $\beta(1,t) = 0$. Then follows

$$\begin{aligned} B(\sigma, t) &= max(\{B(\sigma, i, t) | 1 \leq i \leq \ell(t), B_{\beta(\sigma,i,t)}(i,t) \neq 0\} \cup \{0\}) \\ &= 0 = Depth(t) = Depth(\hat{\sigma}[t]). \end{aligned}$$

Assume that $Depth(\hat{\sigma}[t]) = B(\sigma, t)$ if $Depth(t) < p$ for a natural number $p \geq 1$. If $Depth(t) = p$ then there are terms $t_1, \cdots, t_{n_i}$ and an $n_i$ary operation symbol f_i with $t = f(t_1, \cdots, t_{n_i})$ and $Depth(t_k) < p$ for $k = 1, \cdots, n_i$. Then we have

$$\begin{aligned} Depth(\hat{\sigma}[t]) &= Depth(S_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_{n_i}])) \\ &= max\{Depth_j(\sigma(f_i)) + Depth(\hat{\sigma}[t_j]) | 1 \leq j \leq n_i, x_j \in var(\sigma(f_i))\} \\ &\quad \text{(by Theorem 3.3)} \\ &= max\{Depth_j(\sigma(f_i)) + B(\sigma, t_j) | 1 \leq j \leq n_i, x_j \in var(\sigma(f_i))\} \\ &\quad \text{(by our hypothesis)} \qquad (*) \\ &= max\{Depth_j(\sigma(f_i)) + max(\{B(\sigma, i, t_j) | 1 \leq i \leq \ell(t_j), \\ &\quad B_{\beta(i,t_j)}(\sigma, i, t_j) \neq 0\} \cup \{0\})\} | 1 \leq j \leq n_i, x_j \in var(\sigma(f_i)))\}. \end{aligned}$$

If $n(j), 1 \leq j \leq n_i$ denotes the length of t_j, then we define:

$$q := \sum_{j=1}^{n_i} n(j), \quad \delta_j := \sum_{a=1}^{j} n(a) \quad \text{and} \quad \beta_j = q - \sum_{j=1}^{n_i} n(a).$$

It is easy to check that $\beta(i,t) = \beta(i - \delta_j, t_j) + 1$ for $1 \leq j \leq n_i, \delta_j < i \leq \beta_j$ and $B_a(\sigma, i, t) = B_a(\sigma, i - \delta_j, t_j)$ for $1 \leq j \leq n_i, \delta_j < i \leq \beta_j$ and $0 \leq a \leq \beta(i - \delta_j, t_j)$. Since

$$A_{\beta(i,t)}(i,t) = t = f_i(t_1, \cdots, t_{n_i}), \quad A_{\beta(i-\delta_j, s_j)}(i,t) = t_j$$

for $\delta_j < i \leq \beta_j$ and $1 \leq j \leq n_i$ we have $B_{\beta(i,t)}(\sigma, i, t) = Depth_j(\sigma(f_i))$ for $\delta_j < i \leq \beta_j$ and $1 \leq j \leq n_i$. Hence for $1 \leq j \leq n_i$ and $1 \leq i \leq n(j)$ there holds

$$B(\sigma, i, t_j) = \sum_{a=0}^{\beta(i,t_j)} B_a(\sigma, i, t_j) = \sum_{a=0}^{\beta(i+\delta_j,t)-1} B_a(\sigma, i + \delta_j, t).$$

Now we continue with (*) and obtain by substitution

$$Depth(\hat{\sigma}[t]) =$$

$$= max(\{B_{\beta(i+\delta_j,t)}(\sigma, i + \delta_j, t) + max(\{ \sum_{a=0}^{\beta(i+\delta_j,t)-1} B_a(\sigma, i + \delta_j, t) | 1 \leq i \leq n(j),$$

$$B_{\beta(i+\delta_j,t)-1}(\sigma, i+\delta_j, t) \neq 0\} \cup \{0\})|1 \leq j \leq n_i, x_j \in var(\sigma(f_i))\}$$

$$= max(\{B_{\beta(i+\delta_j,t)}(\sigma, i+\delta_j, t) + \sum_{a=0}^{\beta(i+\delta_j,t)-1} B_a(\sigma, i+\delta_j, t)$$

$$|1 \leq i \leq n(j), 1 \leq j \leq n_i, x_j \in var(\sigma(f_i)), B_{\beta(i+\delta_j,t)-1}(\sigma, i+\delta_j, t) \neq 0\} \cup \{0\})$$

(since $B_{\beta(i+\delta_j,t)}(\sigma, i+\delta_j, t) = const_j$ for $1 \leq i \leq n(j)$ and $1 \leq j \leq n_i$ for a natural number $const_j$)

$$= max(\{\sum_{a=0}^{\beta(i+\delta_j,t)} B_a(\sigma, i+\delta_j, t)|1 \leq i \leq n(j), 1 \leq j \leq n_i,$$

$$B_{\beta(i+\delta_j,t)-1}(\sigma, i+\delta_j, t) \neq 0\} \cup \{0\})$$

$$= max(\{B(\sigma, i+\delta_j, t)|1 \leq i \leq n(j), 1 \leq j \leq n_i, B_{\beta(i+\delta_j,t)-1}(\sigma, i+\delta_j, t) \neq 0\} \cup \{0\})$$

$$= max(\{B(\sigma, i, t)|1 \leq i \leq \ell(t), B_{\beta(i,t)}(i, t) \neq 0\} \cup \{0\}) = B(\sigma, t).$$

■

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